Typing the Numeric Tower

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PLT

PADL 2012 - January 24th, 2012

Approaches to numerics

- Traditional

  Java, C(++), Fortran, ...

- Type Classes

  Haskell, Clean, ...

- Numeric Tower

  Racket, Scheme, Smalltalk, ...

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$\}$ Typed Numeric Tower

Typed Racket

Our criteria

- Ease of expression
- Domain fidelity
- Static checking
- Performance

Our benchmarks

$$x_{i+1} \equiv A \cdot x_i \pmod{p}$$

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

Type Classes

$$x_{i+1} \equiv A \cdot x_i \pmod{p}$$

```
p = (2^31) - 1
a = 7^5
x = unsafePerformIO $ newIORef 42

genRandom min max
  = do old <- readIORef x
       writeIORef x (mod (a * old) p)
       new <- readIORef x
       return $ min + (((max-min) * (fromInteger new))
                       / (fromInteger p))
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genRandom 2.3 7.4
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genRandom (toRational 2) (toRational 7)
  -- => 9927678409 % 2147483647
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Ease of expression ✗

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$$x_{i+1} \equiv A \cdot x_i \pmod{p}$$

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(define A (expt 7 5))
(define x 42) ; state of the PRNG

(define (gen-random min max)
  (set! x (modulo (* A x) p))
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q 2.4 36.2 (-7.5)
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Static checking	✗
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	Type Classes		Numeric Tower	
	PRNG	Quad	PRNG	Quad
Ease of expression	✗	✗	✓	✓
Domain fidelity	✓	✗	✓	✓
Static checking	✓	✓	✗	✗
Performance	✓	✓	✗	✗

	Type Classes		Numeric Tower		Typed Numeric Tower
	PRNG	Quad	PRNG	Quad	All programs
Ease of expression	✗	✗	✓	✓	✓
Domain fidelity	✓	✗	✓	✓	✓
Static checking	✓	✓	✗	✗	✓
Performance	✓	✓	✗	✗	✓

Typed Numeric Tower

Ease of expression	✓
Domain fidelity	✓
Static checking	✗
Performance	✗

Typed Numeric Tower

Ease of expression	✓	
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Static checking	✓	Powerful type system
Performance	✗	

Typed Numeric Tower

Ease of expression	✓	
Domain fidelity	✓	
Static checking	✓	Powerful type system
Performance	✓	Type-driven optimization

The Type System

- Union types
- Function intersection types
- Occurrence typing

Union types

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302 : Integer
302 : (U Integer Float)
3.2 : (U Integer Float)
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23.2 : Positive-Float
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  ```
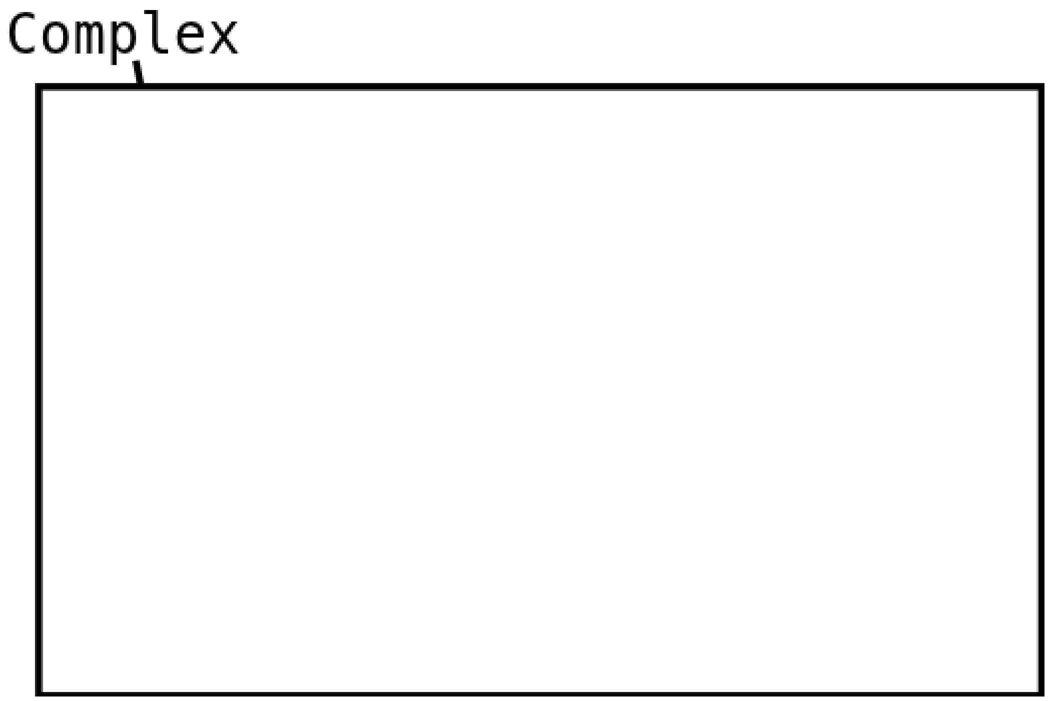
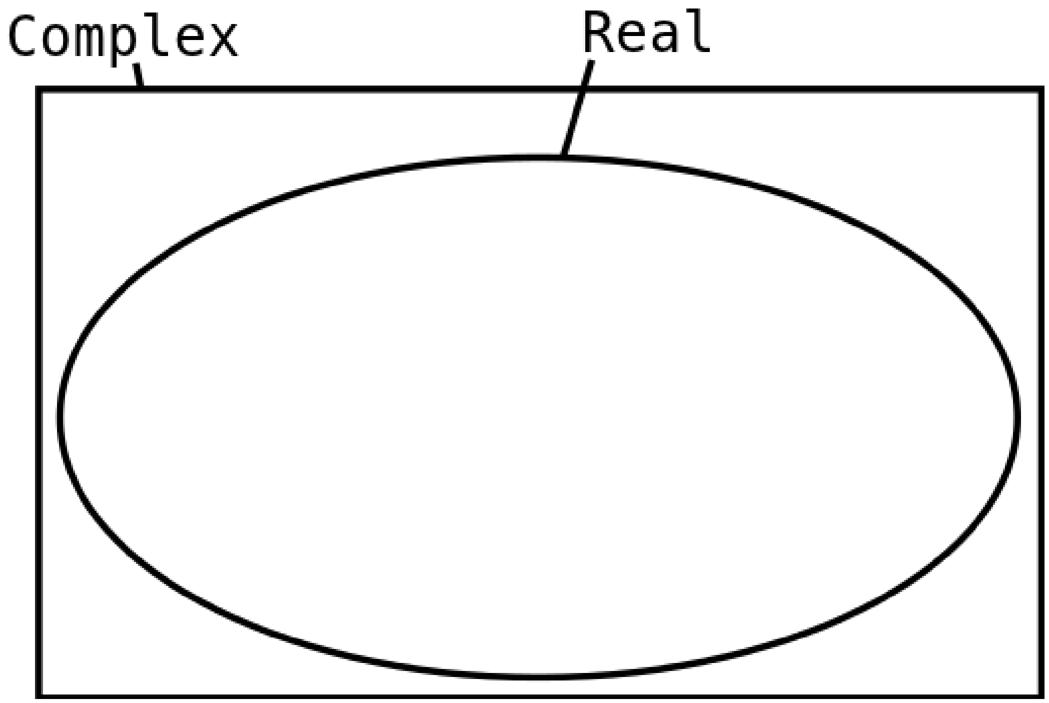

  Representation independence
  Performance ✓

Union types



Union types



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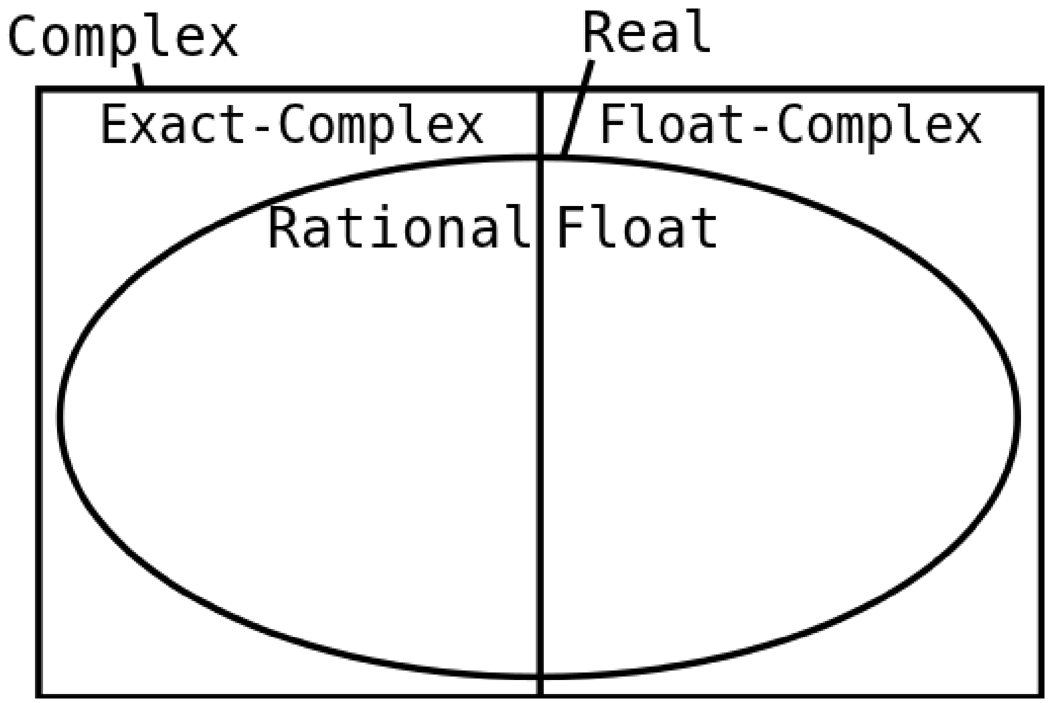

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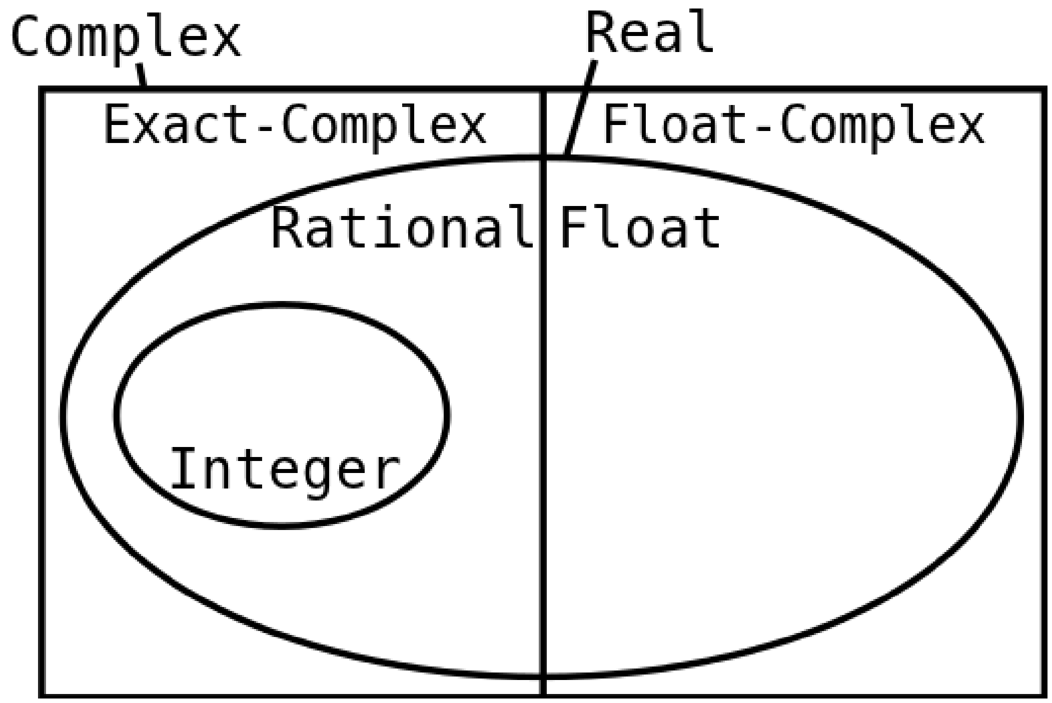
Complex
Real
Exact-Complex
Float-Complex
Rational
Float
Integer

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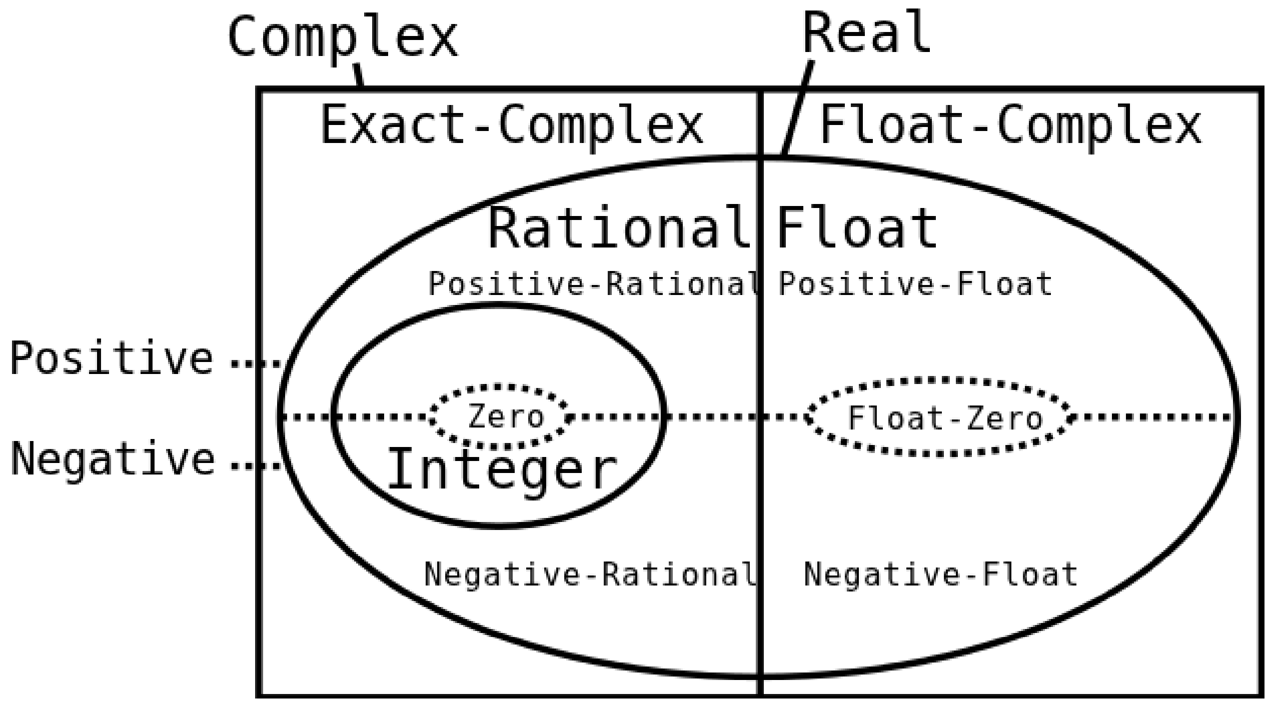
Complex
Real
Exact-Complex
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Float
Positive-Rational
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Zero
Float-Zero
Negative
Integer
Negative-Rational
Negative-Float

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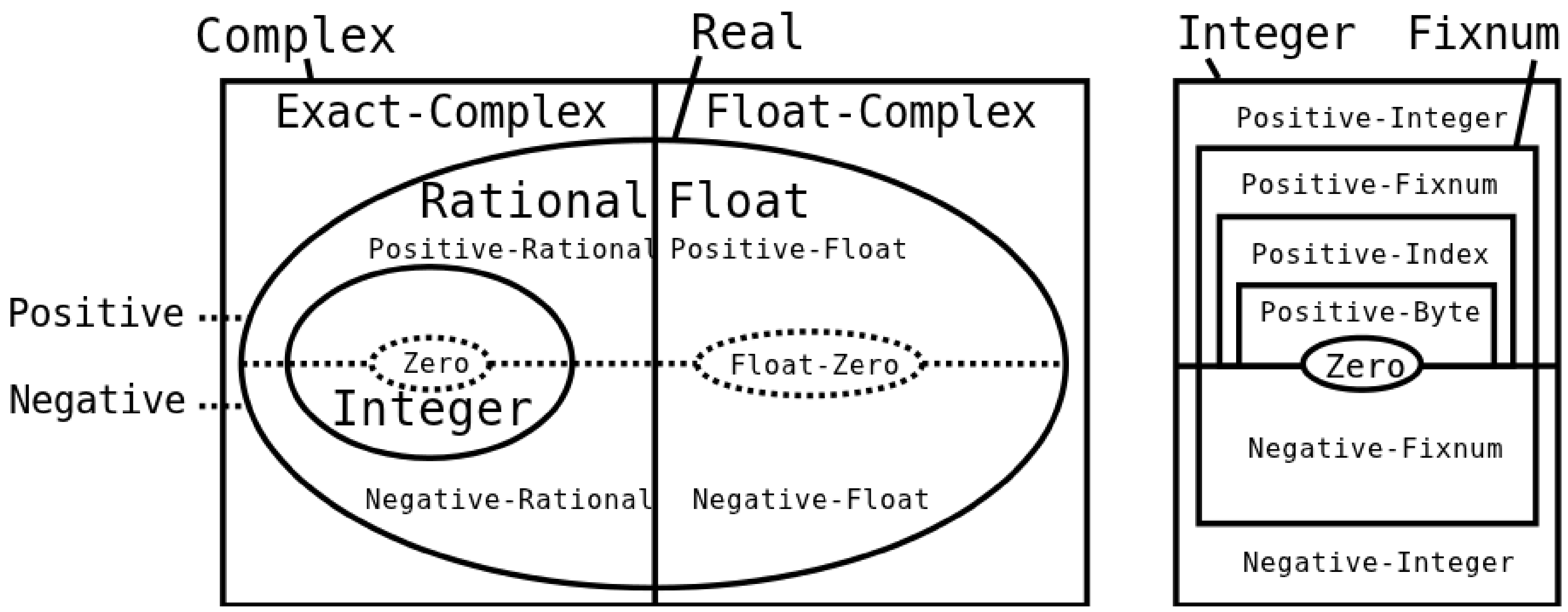
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Fixnum
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Positive-Fixnum
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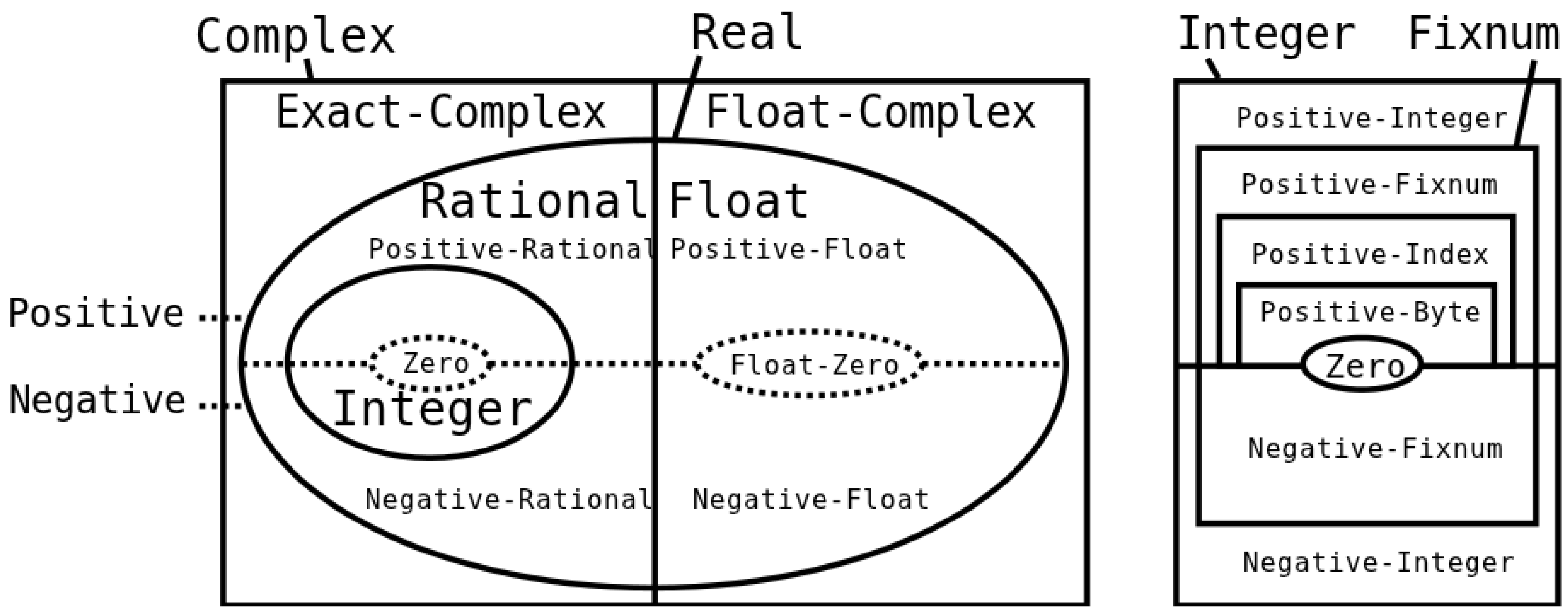

Performance ✓

Function intersection types

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               …))
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$$> \; : \; \left(x : \texttt{Real} \;\; y : \texttt{Nonnegative-Real} \xrightarrow{\texttt{Positive}_x} \texttt{Boolean}\right)$$

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- Arithmetic specialization
- Unboxing
- Arity raising of complex numbers
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Positive-Float <: Float
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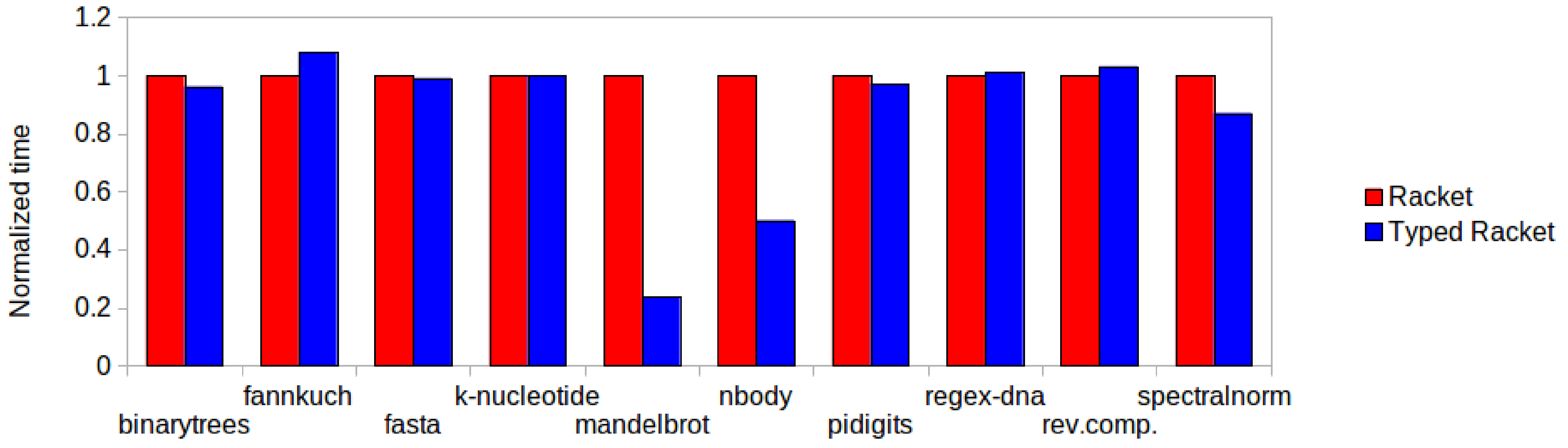

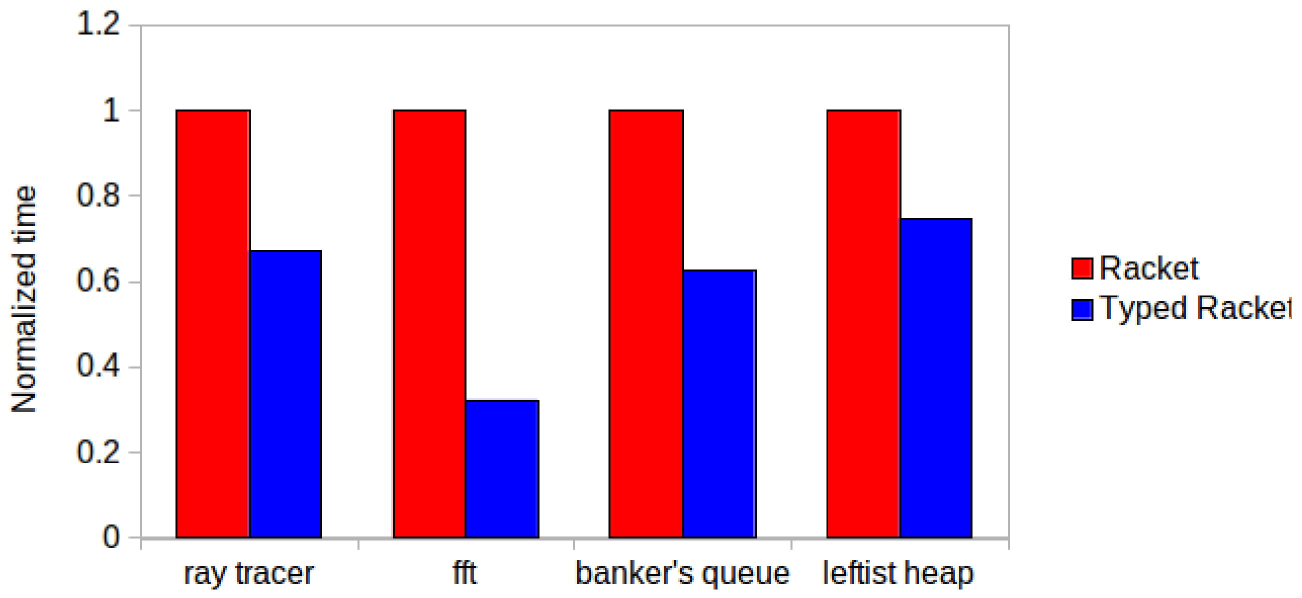

Smaller is better

Usability

```
> (+ 1 "A")

Type Checker: No function domains
  matched in function application:
Domains:
  Zero Zero
  Zero Positive-Byte
  Byte Positive-Byte
  Byte Byte
... <snip 58 lines> ...
  Real Real
  Float-Complex Number
  Number Float-Complex
  Number Number
Arguments: Positive-Byte String
 in: (+ 1 "A")
```

```
[+ (from-cases
    (binop -Zero)
    (map (lambda (t) (commutative-binop t -Zero t))
         (list -One -PosByte -Byte -PosIndex -Index
               -PosFixnum -NonNegFixnum -NegFixnum -NonPosFixnum -Fixnum))
    (-> -PosByte -PosByte -PosIndex)
    (-> -Byte -Byte -Index)
    (-> -PosByte -PosByte -PosByte -PosIndex)
    (-> -Byte -Byte -Byte -Index)
    (commutative-binop -PosIndex -Index -PosFixnum)
    (-> -PosIndex -Index -Index -PosFixnum)
    (-> -Index -PosIndex -Index -PosFixnum)
    (-> -Index -Index -PosIndex -PosFixnum)
    (-> -Index -Index -NonNegFixnum)
    (-> -Index -Index -Index -NonNegFixnum)
    (commutative-binop -NegFixnum -One -NonPosFixnum)
    (commutative-binop -NonPosFixnum -NonNegFixnum -Fixnum)
    (commutative-case -PosInt -Nat -PosInt)
    (commutative-case -NegInt -NonPosInt -NegInt)
    (map varop (list -Nat -NonPosInt -Int))
    (commutative-case -PosRat -NonNegRat -PosRat)
    (commutative-case -NegRat -NonPosRat -NegRat)
    (map varop (list -NonNegRat -NonPosRat -Rat))
    (commutative-case -PosFlonum -NonNegReal -PosFlonum)
    (commutative-case -PosReal -NonNegFlonum -PosFlonum)
    (commutative-case -NegFlonum -NonPosReal -NegFlonum)
    (commutative-case -NegReal -NonPosFlonum -NegFlonum)
    (commutative-case -NonNegFlonum -NonNegReal -NonNegFlonum)
    (commutative-case -NonPosFlonum -NonPosReal -NonPosFlonum)
    (commutative-case -Flonum -Real -Flonum)
    (commutative-case -PosSingleFlonum (Un -NonNegRat -NonNegSingleFlonum) -PosSingleFlonum)
    (commutative-case (Un -PosRat -PosSingleFlonum) -NonNegSingleFlonum -PosSingleFlonum)
    (commutative-case -NegSingleFlonum (Un -NonPosRat -NonPosSingleFlonum) -NegSingleFlonum)
    (commutative-case (Un -NegRat -NegSingleFlonum) -NonPosSingleFlonum -NegSingleFlonum)
    (commutative-case -NonNegSingleFlonum (Un -NonNegRat -NonNegSingleFlonum) -NonNegSingleFlonum)
    (commutative-case -NonPosSingleFlonum (Un -NonPosRat -NonPosSingleFlonum) -NonPosSingleFlonum)
    (commutative-case -SingleFlonum (Un -Rat -SingleFlonum) -SingleFlonum)
    (commutative-case -PosInexactReal -NonNegReal -PosInexactReal)
    (commutative-case -PosReal -NonNegInexactReal -PosInexactReal)
    (commutative-case -NegInexactReal -NonPosReal -NegInexactReal)
    (commutative-case -NegReal -NonPosInexactReal -NegInexactReal)
    (commutative-case -NonNegInexactReal -NonNegReal -NonNegInexactReal)
    (commutative-case -NonPosInexactReal -NonPosReal -NonPosInexactReal)
    (commutative-case -InexactReal -Real -InexactReal)
    (commutative-case -PosReal -NonNegReal -PosReal)
    (commutative-case -NegReal -NonPosReal -NegReal)
    (map varop (list -NonNegReal -NonPosReal -Real -ExactNumber))
    (commutative-case -FloatComplex N -FloatComplex)
    (commutative-case -Flonum -InexactComplex -FloatComplex)
    (commutative-case -SingleFlonumComplex (Un -Rat -SingleFlonum -SingleFlonumComplex) -SingleFlonumComplex)
    (commutative-case -InexactComplex (Un -Rat -InexactReal -InexactComplex) -InexactComplex)
    (varop N))]
```

56 lines of type DSL
22k of printout

```
> (+ 1 "A")

Type Checker: No function domains
  matched in function application:
Domains: Number Number
Arguments: Positive-Byte String
 in: (+ 1 "A")
```

```
(define x (box 3))
```

```
(define x (box 3)) : (Boxof Positive-Byte)
```

```
(define x (box 3)) : (Boxof Positive-Byte)

(set-box! x 2000)          ✗
```

```
(define x (box 3)) : (Boxof Number)

(set-box! x 2000)          ✓
```

```
(define x (box 3)) : (Boxof Number)

(set-box! x 2000)          ✓

(vector-ref v (unbox x))   ✗
```

```
(define x (box 3)) : (Boxof Natural)

(set-box! x 2000)          ✓

(vector-ref v (unbox x))   ✓
```

Typed Numeric Tower

Ease of expression	✓
Domain fidelity	✓
Static checking	✓
Performance	✓

Key type system features

- Union types
- Function intersection types
- Occurrence typing

Typed Numeric Tower

Ease of expression	✓
Domain fidelity	✓
Static checking	✓
Performance	✓

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- Union types
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`racket-lang.org`